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Quantum Corrections to the Entropy of a Black Hole with a Global Monopole

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Quantum corrections to the entropy of a static, spherically symmetric black hole–global monopole system arising from the Dirac spinor field are investigated by using the brick wall method. It is shown that if we ignore the usual contribution from the vacuum surrounding the system, then the quantum corrections for the static black hole consist of two parts: One is a quadratically divergent term which takes a geometric character. The other is a logarithmically divergent term which is not proportional to the area of the horizon. The renormalization of the quadratic and logarithmic divergences is also investigated.

1. INTRODUCTION

It is a remarkable and long-known fact that the classical Bekenstein–Hawking (BH) entropy of a four-dimensional black hole is proportional to the area of the horizon [1, 2], i.e., $S = A_h/(4G)$, where A_h is the area of the horizon and G the gravitational constant. However, it is reasonable to ask whether this geometric character of the hole entropy remains valid when quantum corrections (say, due to quantum fluctuations of the matter field in the black hole background [3, 4]) are taken into account. Recently, much attention has been focused on this problem, and it has been shown [5, 6] that the BH result is only the leading contribution in the integral of an entropy density over a cross section of the horizon, i.e., $S = \oint_H \rho_s \, d^2x$, because $\rho_s = 1/(4G)$ in the BH entropy.

The quantum corrections to the entropy arising from the scalar field have been studied extensively and many interesting results have been obtained. Using the brick wall approach [3], 't Hooft found that the quantum

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corrections to the entropy for the Schwarzchild black hole takes the geometric character $A_h/(48\pi\varepsilon^2)$, where ε is the ultraviolet cutoff. However, using the Euclidean path integral method of Gibbons and Hawking [7], Solodukhin [8] found that in addition to the 't Hooft result, the entropy still has a logarithmically divergent term $S_{\log} = (1/45)\ln(\Lambda/\varepsilon)$, where Λ is the infrared cutoff. Recently, Frolov and Pursaw studied the quantum corrections to the entropy of the static black holes arising from the Dirac spinor field [9]. The purpose of this paper is to investigate the quantum corrections to the entropy of the black hole with a global monopole arising from the Dirac spinor field by using the brick wall approach [3]. We begin in Section 2 by considering the Dirac spinor field in this black hole background. We introduce a certain transformation to reduce the Dirac field equations to ordinary differential equations. In Section 3 we use 't Hooft's approach to calculate the black hole entropy. In the final section we discuss the renormalization of the entropy divergences.

It is known that certain gauge theories allow the possibility of topological defects, such as global monopoles and cosmic strings, and these defects represent objects which might have been created in the early universe [10]. The appearance of global monopoles and cosmic strings may have an important cosmological consequence: they can produce observational effects, and may also affect galaxy formation [10–12]. In particular, for a Schwarszchild black hole with a global monopole inside, it has been shown that the metric outside the black hole can be written as [13–15]

$$ds^2 = \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - 8\pi\eta^2 - 2M/r} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{1}$$

where M is the Schwarzschild mass parameter and η is the scale of symmetry breaking. The classical thermodynamics of the black hole (1) has been studied by Aryal [13] and Jing *et al.* [14], whereas its quantum corrections to entropy arising from the scalar field also has been studied by us [16].

2. DIRAC FIELD EQUATIONS

In curved space-time, the massless Dirac spinor field equations read [17]

$$\begin{aligned}
(D + \varepsilon - \rho)F_1 + (\bar{\delta} + \pi - \alpha)F_2 &= 0\\
(\Delta + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 &= 0\\
(D + \varepsilon^* - \rho^*)G_2 - (\delta + \pi^* - \alpha^*)G_1 &= 0\\
(\Delta + \mu^* - \gamma^*)G_1 - (\bar{\delta} + \beta^* - \tau^*)G_2 &= 0
\end{aligned} \tag{2}$$

where $\Phi = (F_1,F_2,G_1,G_2)^T$ is the four-component spinor, D, Δ, δ, and $\bar{\delta}$ are ordinary differential functors, and α, β, γ, . . . are Newman–Penrose spin coefficients, which are defined by [18]

$$\begin{aligned} &D = l^\mu\partial_\mu, \qquad \Delta = n^\mu\partial_\mu, \qquad \delta = m^\mu\partial_\mu, \qquad \bar{\delta} = \bar{m}^\mu\partial_\mu \\ &\rho = l_{\mu;\nu}m^\mu\bar{m}^\nu, \qquad \tau = l_{\mu;\nu}m^\mu n^\nu, \qquad \mu = -n_{\mu;\nu}\bar{m}^\mu m^\nu, \qquad \pi = -n_{\mu;\nu}\bar{m}^\mu l^\gamma \\ &\alpha = 1/2(l_{\mu;\nu}n^\mu\bar{m}^\nu - m_{\mu;\nu}\bar{m}^\mu\bar{m}^\nu), \qquad \beta = 1/2(l_{\mu;\nu}n^\mu m^\nu - m_{\mu;\nu}\bar{m}^\mu m^\nu) \\ &\gamma = 1/2(l_{\mu;\nu}n^\mu n^\nu - m_{\mu;\nu}\bar{m}^\mu n^\nu), \qquad \varepsilon = 1/2(l_{\mu;\nu}n^\mu l^\nu - m_{\mu;\nu}\bar{m}^\mu l^\nu) \end{aligned} \tag{3}$$

where the null tetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ satisfies the null vector, pseudoorthogonality, and metric relations

$$\begin{aligned} &l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = \bar{m}^\mu\bar{m}_\mu = l^\mu m_\mu = l^\mu\bar{m}_\mu = n^\mu m_\mu = n^\mu\bar{m}_\mu = 0 \\ &l^\mu n_\mu = -\, m^\mu\bar{m}^\mu = 1 \\ &g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu\bar{m}_\nu - \bar{m}_\mu m_\nu \end{aligned} \tag{4}$$

Now, we will reduce Eqs. (2) in the above black hole space-time (1). According to the conditions (4) and the metric (1), we can take the null tetrad vector as

$$\begin{aligned} l_\mu &= \delta^0_\mu - (1 - 8\pi\eta^2 - 2M/r)^{-1}\, \delta^2_\mu \\ n_\mu &= 1/2[(1 - 8\pi\eta^2 - 2M/r)\, \delta^0_\mu + \delta^1_\mu] \\ m_\mu &= -r/\sqrt{2}(\delta^2_\mu + i \sin\theta\, \delta^3_\mu) \\ \bar{m} &= -r/\sqrt{2}(\delta^2_\mu - i \sin\theta\, \delta^3_\mu) \end{aligned} \tag{5}$$

Consequently, the differential functor and the coefficients (3) are given by

$$\begin{aligned} &D = (1 - 8\pi\eta^2 - 2M/r)^{-1}\, \partial_t + \partial_r \\ &\Delta = 1/2[\partial_t - (1 - 8\pi\eta^2 - 2M/r)\partial_r] \\ &\delta = (\sqrt{2}r)^{-1}\, (\partial_\theta + ir \sin^{-1}\theta\, \partial_\varphi), \qquad \bar{\delta} = (\sqrt{2}r)^{-1}(\partial_\theta - i \sin^{-1}\theta\, \partial_\varphi) \\ &\alpha = -\beta = -(2\sqrt{2}r)^{-1}\, \mathrm{ctg}\, \theta, \qquad \gamma = M/(2r^2), \qquad \rho = -r^{-1} \\ &\mu = -(1 - 8\pi\eta^2 - 2M/r)/(2r), \qquad \varepsilon = \tau = \pi = 0 \end{aligned} \tag{6}$$

In order to reduce the Dirac field equations (2), we introduce the following transformations [19] on the Dirac field functions Φ:

$$\begin{aligned} F_1 &= r^{-1}R_{-1/2}(r)\Theta_{-1/2}(\theta)\exp(-iEt + im\varphi) \\ F_2 &= R_{+1/2}(r)\Theta_{+1/2}(\theta)\exp(-iEt + im\varphi) \end{aligned} \tag{7}$$

$$G_1 = R_{+1/2}(r)\Theta_{-1/2}(\theta)\exp(-iEt + im\varphi)$$
$$G_2 = r^{-1}R_{-1/2}(r)\Theta_{+1/2}(\theta)\exp(-iEt + im\varphi)$$

Substituting the Eqs. (6) and (7) into the Dirac field equations (2) then yields

$$r^2\left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)\frac{d^2R_{+1/2}}{dr^2} + 3r\left(1 - 8\pi\eta^2 - \frac{M}{r}\right)\frac{dR_{+1/2}}{dr} + \left[\frac{E^2r^2}{(1 - 8\pi\eta^2 - 2m/r)} + iE\left(r - \frac{M}{1 - 8\pi\eta^2 - 2M/r}\right) + 1 - 8\pi\eta^2 - \lambda^2\right]R_{+1/2} = 0 \tag{8}$$

$$r^2\left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)\frac{d^2R_{-1/2}}{dr^2} + r\left(1 - 8\pi\eta^2 - \frac{M}{r}\right)\frac{dR_{-1/2}}{dr} + \left[\frac{E^2r^2}{(1 - 8\pi\eta^2 - 2m/r)} - iE\left(r - \frac{M}{1 - 8\pi\eta^2 - 2M/r}\right) - \lambda^2\right]R_{-1/2} = 0 \tag{9}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{m^2}{\sin^2\theta} + \lambda^2\right]\Theta_{\pm1/2} + \left(\frac{1}{4}\,\mathrm{ctg}^2\theta\ \mu\ \frac{m\cos\theta}{\sin^2\theta} - \frac{1}{2\sin^2\theta}\right)\Theta_{\pm1/2} = 0 \tag{10}$$

where the Eqs. (8)–(9) and (10) are the radial and angular equations, respectively. Note that one can modify Eq. (10) into Legendre form by using the appropriate angular coordinate transformation. So we have $\lambda^2 = l(l+1)$, and its angular solution $Y_{lm}(\theta, \varphi) = \Theta_{\pm1/2}(\theta)\exp(im\varphi)$. Thus the four-component wave function of the Dirac field in the static spherically symmetric black hole background with the metric (1) is given by

$$\Phi = [F_1, F_2, G_1, G_2]^T \rightarrow [r^{-1}R_{-1/2}, R_{+1/2}, R_{+1/2}, r^{-1}R_{-1/2}]^T\,Y_{lm}(\theta, \varphi) \times \exp(-iEt) \tag{11}$$

3. ENTROPY CALCULATION

In the following we will use the brick wall method of 't Hooft to study quantum corrections to the entropy arising from the Dirac spinor fields for the black hole described by Eq. (1). In the brick wall method [3], the wave function in Eqs. (8) and (9) is cut off just outside the horizon, i.e., $R_{+1/2} = R_{-1/2} = 0$ at $r = r_h + h$, where h is a small, positive ultraviolet cutoff, $r_h = 2M/(1 - 8\pi\eta^2)$, the horizon [20]. To eliminate the infrared divergences, a second cutoff is introduced at a large radius $L \gg r_h$: $R_{+1/2} = R_{-1/2} = 0$ for $r \geq L$. Using the Webtzel–Kramers–Brillouin approximation with $R_{\pm 1/2} = e^{iS_{\pm 1/2}(r)}$, we find that Eqs. (8) and (9) yield the radial wave number $k(r, l, E) \equiv \partial_r S$:

$$k_{+1/2} = \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)^{-1} \times \left\{E^2 + \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)\frac{1 - 8\,\pi\eta^2 - l(l+1)}{r^2}\right\}^{1/2} \tag{12}$$

$$k_{-1/2} = \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)^{-1}\left\{E^2 - \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)\frac{l(l+1)}{r^2}\right\}^{1/2}$$

The number of modes with energy not exceeding E is determined by summing over the degeneracy of the angular modes, and finding the radial mode number by counting the number of nodes in the radial wave function:

$$g_{\pm 1/2}(E) \equiv \int dl\,(2l+1)\int_{r_h+h}^{L} dr\,\frac{1}{\pi}\,k_{\pm 1/2}\,(r, l, E) \tag{13}$$

where the sum over the angular quantum number l has been approximated by an integral and the integral is taken only over those values for which the square root exists.

To get the thermodynamic properties of the system we consider the free energy of a thermal ensemble of Dirac spinor particles with an inverse temperature β, i.e., $F = -\beta^{-1}\sum_N \ln\,(1 + e^{-\beta E_N})$. Using Eqs. (12) and (13) to determine the density of states and ignoring the usual contribution from the vacuum surrounding the system as 't Hooft did [3], we obtain the main contribution of the free energy

$$F = -\frac{2}{\pi}\int_0^\infty \frac{dE}{e^{\beta E}+1}\int_{r_h+h}^{L} dr\left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)^{-1}\int dl\,(2l+1) \times \left\{\left[E^2 + \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)\frac{1 - 8\pi\eta^2 - l(l+1)}{r^2}\right]^{1/2} + \left[E^2 - \left(1 - 8\pi\eta^2 - \frac{2M}{r}\right)\frac{l(l+1)}{r^2}\right]^{1/2}\right\}$$

$$\approx -\frac{7\pi^3}{45b^2h}\left(\frac{r_h}{\beta}\right)^4 - \frac{28\pi^3 r_h^3}{45b^2\beta^4}\ln\left(\frac{L}{h}\right) \tag{14}$$

where $b = 1 - 8\pi\eta^2$, so the entropy will be

$$S^q = \beta^2\,\partial_\beta F = \frac{7r_h}{720\sqrt{bh}} + \frac{7}{180\sqrt{b}}\ln\left(\frac{L}{h}\right) \tag{15}$$

where $\beta = 8\pi M/(1 - 8\pi\eta^2)^{3/2}$ [20]. Note that the proper distance from r_h to $r_h + h$ is given by

$$\delta = \int_{r_h}^{r_h+h}\sqrt{g_{rr}}\,dr \approx 2\sqrt{\frac{r_h h}{b}},$$

We set $\delta^2 = 2\varepsilon^2/15$ and $\Lambda^2 = L\varepsilon^2/h = 30r_hL/b$, where ε is the ultraviolet cutoff and Λ is the infrared cutoff [16]; then the quantum corrections to the entropy of the black hole described by Eqs. (1) can be written as

$$S^q = \frac{7A_h}{96\pi b^{3/2}\varepsilon^2} + \frac{7}{90\sqrt{b}}\ln\left(\frac{\Lambda}{\varepsilon}\right) \tag{16}$$

where $A_h = 4\pi\gamma_h^2$ is the horizon area of the black hole. The entropy (16) contains two parts: The first part is an intrinsic contribution from the horizon and it diverges quadratically as $\varepsilon \to 0$, which takes a geometric character; the second part is logarithmically divergent, is not proportional to the horizon area, and depends on the black hole characteristics and therefore cannot be regarded as a nonessential additive constant and neglected.

4. RENORMALIZATION OF THE ENTROPY

In conclusion, using the brick wall method of 't Hooft, we studied the entropy of the quantum Dirac field in the black hole space-time. The black hole with a monopole was considered. If we ignore the contribution from the vacuum surrounding the system and take $\delta^2 = 2\varepsilon^2/15$ and $\Lambda^2 = L\varepsilon^2/h$, the result consists of two parts. One is the intrinsic contribution from the horizon, which diverges quadratically on the horizon. This part takes the geometric character $7\,A_h/96\pi b^{3/2}\varepsilon^2$, which can be regarded as a renormalization of the gravitational constant determined by $1/G_{\text{ren}} = 1/G + 7/(24\pi b^{3/2}\varepsilon^2)$ as discussed in refs. 21–23. The other part is the logarithmically divergent term, which is not proportional to the horizon area and in general depends on the black hole characteristics. Therefore, the term in Eq. (16) cannot be regarded as a nonessential additive constant and neglected. If we define the quantum-corrected radius of the horizon $r_{h,\text{ren}}^2 = r_h^2 + \xi l_{Pl}^2$ (where $l_{Pl}^2 =$

G_{ren} is the Planck length), the quantity $\xi = \zeta(M, \text{etc.}) \ln(\Lambda/\varepsilon)$ absorbs the logarithmic divergence of the entropy. The parameter for the black hole with a global monopole is given by $7/(90\sqrt{1 - 8\pi\eta^2})$.

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